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Comparison of Iterative and Direct Solution Methods for Viscous Flow Problems

C. P. van Dam,* and M. Hafez†
University of California, Davis, Davis, California

Introduction

THE purpose of this study is to compare quantitatively several numerical solution techniques for solving two-dimensional viscous flow problems, including separated flows, at moderate to high Reynolds numbers. These solution techniques have various degrees of implicitness, and they are compared on the basis of convergence, CPU time, storage requirements, and solution accuracy. The stream-function (ψ) vorticity (ω) approach is used to formulate the partially parabolized Navier-Stokes (PPNS) equations that describe the two-dimensional incompressible flow. The PPNS equations are quite similar to the Navier-Stokes (NS) equations; streamwise diffusion is the only physical process neglected, and terms representing this diffusion are dropped from the NS equations. The governing equations are

$$\alpha\omega_t + (u\omega)_x + (v\omega)_y = (1/R)\omega_{yy} \tag{1}$$

$$-\omega = \psi_{xx} + \psi_{yy} \tag{2}$$

where the term $\alpha\omega_t$ represents an artificial time-dependent term, and the velocity $u = \psi_y$ and $v = -\psi_x$. At the solid wall, the no-slip boundary condition requires $u = 0$ and $v = 0$. For external flows $\omega = 0$ and $u = U(x)$ at the upper boundary. For internal flows the boundary conditions at the centerline of, e.g., a channel, are $\omega = 0$ and $\psi = \psi(x_0, y_c)$. At the inflow boundary ψ and ω are prescribed, whereas at the outflow boundary the PPNS equations are reduced to the boundary-layer equations by neglecting the ψ_{xx} term.

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*Assistant Professor, Division of Aeronautical Science and Engineering, Department of Mechanical Engineering. Member AIAA.

†Professor, Division of Aeronautical Science and Engineering, Department of Mechanical Engineering. Associate Fellow AIAA.

Solution Techniques

Method I

The first solution technique employs a simple space marching procedure for the coupled stream-function and vorticity equations. Several authors have developed similar iterative methods to solve the NS equations or reduced equations.[1-5] The vorticity equation is linearized by lagging its coefficients. The resulting equation plus the stream-function equation are solved simultaneously for ω and ψ by using a line-relaxation procedure. Multiple global sweeps are required to obtain convergence. Centered differencing is applied for all terms except for the convective term $(u\omega)_x$. The problem of loss of diagonal dominance forces the use of an upwind differencing scheme for this term. Previously, the authors used a conservative first-order upwing differencing scheme for the convective term in the streamwise direction.[6] However, this scheme required excessively fine meshes to obtain grid-independent solutions. In this study the following second-order upwind differencing scheme is applied:

$$(u\omega)_x = \frac{3u_{i,j}\omega_{i,j} - 4u_{i-1,j}\omega_{i-1,j} + u_{i-2,j}\omega_{i-2,j}}{2\Delta x}$$

$$\text{if} \quad u_{i+\frac{1}{2},j} \geq 0 \quad \text{and} \quad u_{i-\frac{1}{2},j} \geq 0 \tag{3a}$$

$$(u\omega)_x = \frac{-3u_{i,j}\omega_{i,j} + 4u_{i-1,j}\omega_{i-1,j} - u_{i-2,j}\omega_{i-2,j}}{2\Delta x}$$

$$\text{if} \quad u_{i+\frac{1}{2},j} < 0 \quad \text{and} \quad u_{i-\frac{1}{2},j} < 0 \tag{3b}$$

$$(u\omega)_x = \frac{u_{i+1,j}\omega_{i+1,j} - u_{i-1,j}\omega_{i-1,j}}{2\Delta x}$$

$$\text{if} \quad u_{i+\frac{1}{2},j} \geq 0 \quad \text{and} \quad u_{i-\frac{1}{2},j} < 0 \tag{3c}$$

$$(u\omega)_x = \frac{-u_{i+2,j}\omega_{i+2,j} + 3u_{i+1,j}\omega_{i+1,j} - 3u_{i-1,j}\omega_{i-1,j} + u_{i-2,j}\omega_{i-2,j}}{2\Delta x}$$

$$\text{if} \quad u_{i+\frac{1}{2},j} < 0 \quad \text{and} \quad u_{i-\frac{1}{2},j} \geq 0 \tag{3d}$$

It can be shown that this scheme is conservative and that it is no more dissipative than a central differencing scheme. The resulting algebraic expressions have the following form:

$$a_1\psi_{i,j-1} + a_2\psi_{i,j} + a_3\omega_{i,j} + a_4\psi_{i,j+1} = a_5 \tag{4}$$

$$b_1\omega_{i,j-1} + b_2\omega_{i,j} + b_3\omega_{i,j+1} = b_4 \tag{5}$$

At each x location (i = const), the coefficients form a five-diagonal matrix. The solid wall and far-field boundary conditions are implemented as follows:

$$\psi_{i,1} = 0 \tag{6}$$

$$\omega_{i,1} = 2\left[\frac{\psi_{i,1} - \psi_{i,2}}{(\Delta y)^2}\right] \tag{7}$$

$$\omega_{i,N} = 0 \tag{8}$$

$$\frac{\psi_{i+1,N} + \psi_{i-1N}}{2(\Delta x)^2} - \psi_{i,N}\left[\frac{1}{(\Delta y)^2} + \frac{1}{(\Delta x)^2}\right] + \frac{\psi_{i,N-1}}{(\Delta y)^2}$$

$$= -\frac{\omega_{i,N}}{2} - \frac{U(x)}{\Delta y} \tag{9}$$

where M and N represent the number of grid points in the streamwise and normal direction, respectively. The system of equations is solved with a scalar pentadiagonal matrix solver. The convergence of this solution technique can be accelerated by the introduction of a relaxation parameter β for the stream-function equation.[4]

Method II

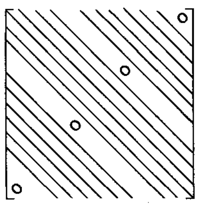
For the second solution procedure, the vorticity transport equation and the stream-function equation are solved separately instead of simultaneously. The discretization is identical to the previous method. First the parabolic vorticity equation is marched line by line using a tridiagonal matrix solver. The wall and far-field boundary conditions provided by Eqs. (7) and (8), respectively, are used. Secondly, the Poisson equation for the stream function is solved for ψ using a standard direct solver.[7] The use of a direct solver for the stream-function equation has been advocated by Ghia et al.[8] for both steady and unsteady flows. The system of equations that is obtained through discretization of the stream-function equation [together with the boundary conditions (6) and (9)] is decomposed into an upper and a lower triangular-banded matrix such that $A = LU$. This LU decomposition for the stream-function equation has to be conducted only once. The process is repeated until convergence is obtained.

Method III

The third solution technique is a "hybrid" procedure, a combination of the previous two methods. The line-relaxation procedure of Method I accelerates the nonlinear interaction between ψ and ω, whereas the use of an elliptic solver for ψ (ω is frozen), as is done in Method II, accelerates the spread of the viscous effects. First, ψ and ω are solved simultaneously line by line using the scalar pentadiagonal matrix solver. Next, after each marching sweep, ψ is updated using the direct solver.

Method IV

In this case the coupled stream-function equation and the vorticity transport equation are solved simultaneously with the direct solver for ψ and ω. All of the terms in both equations, including the convective terms, are centrally differenced. In Eq. (1), the stream-function derivatives with respect to x and y are substituted for the velocities v and u, respectively, and Newton's method is used to linearize the difference form of this equation. The left-hand side (LHS) of the resulting linear system of equations has the following form:

$As =$  $\begin{bmatrix} \delta\psi_{1,1} \\ \delta\omega_{1,1} \\ \delta\psi_{1,2} \\ \vdots \\ \delta\omega_{M,N} \end{bmatrix}$ (10)

This coupled set of equations is solved for $\delta\psi$ and $\delta\omega$ with a direct solver, which uses Gaussian elimination with partial pivoting for stability to compute the LU factorization of matrix A. Iteration is still required to deal with the nonlinearity of the vorticity transport equation. Unfortunately, the LU decomposition must be performed at each iteration because the coefficients of matrix A are a function of ψ and ω.

Method V

For the last method, centered differences are used for all terms of the stream-function and vorticity equations. However, the two equations are solved successively instead of simultaneously. The stream-function equation is solved by a direct solver (see Method II). Next, the vorticity transport equation is also solved using a direct solver. This process is repeated until convergence is obtained.

Storage Requirements

The minimum memory requirement is the storage needed for the two arrays ψ and ω, each of dimensions (M,N). The storage requirement for Method I is about $2MN + 12N$. Using a direct solver clearly increases the demand for storage; Methods II, III, and V require a maximum storage ($M > N$) or about $MN(3N + 6)$. The combination of Newton's method and direct solver for both equations requires the largest memory, approximately $MN(12N + 24)$ for Method IV.

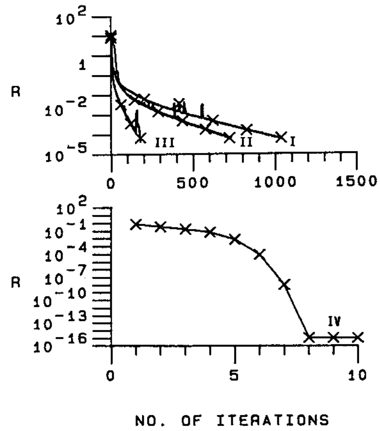


Fig. 1 Maximum absolute residuals of vorticity transport equation for separated flat plate flow using Method I ($\alpha = 0$), Method II ($\alpha = 1.5$), Method III ($\alpha = 0.2$), and Method IV ($\alpha = 0$). Uniform 61×31 grid and $R = 1024$.

Results and Discussion

The following numerical experiments have been conducted on a VAX 11/785 computer using double precision (approximately 16 decimal digits). First, the model problem of a separated flow over a flat plate is analyzed.[9] The inflow and outflow boundaries of the computational domain are located at $x = 1.0$ and $x = 10.0$, respectively. The Reynold's number is based on the reference length ($x = 1.0$) and the freestream velocity. At the inflow boundary, ψ and ω are obtained from boundary-layer calculations. The external velocity U reduces linearly with x from $U(0) = 1.0$ until $U(3.992) = 0.80$, after which U is constant. The wall is located at $y = 0$, and the upper boundary is at $y = 0.75$. For the entire flowfield, the initial condition is taken to be the same as the inflow boundary condition. Convergence is assumed if the maximum absolute residuals of the stream-function equation and the vorticity equation are an order of magnitude less than the truncation error. For Method I, convergence is obtained after 1039 global iterations and 46.1 min of CPU time (Fig. 1). The use of the relaxation parameter β for Method I reduces the number of iterations and the CPU time by a maximum amount of 22% ($\beta = 0.9$). Uncoupling the two equations results in a stability problem for Method II. An artificial time-dependent term is introduced and the minimum value for $\alpha = 1.5$. Use of Method II results in convergence after 729 iterations and 43.1 CPU min (Fig. 1). In Fig. 1, the convergence history for Method III shows that the solution is attained after only 179 iterations and 14.7 min of CPU time. The strong influence of coupling the two equations can be observed by comparing the convergence behavior for Methods IV and V. Quadratic convergence is obtained, and only 7 iterations (30.3 CPU min) are required for the solution in the case of Method IV (Fig. 1). The convergence history for Method V is identical to that of Method II except for a slight reduction in the number of iterations (585) and a large increment in CPU time as a result of the application of the direct solver for the vorticity equation. In Fig. 2, the effect of the discretization technique is shown, and the results are compared with Briley's results.[9] The upwind-differencing schemes produce identical results as the central-differencing schemes, and both sets of results are in good agreement with Briley's solution except for the bubble width.

The model problem of a separated flow in a symmetric diffuser is also analyzed.[4] The comparison of the various methods in terms of rate of convergence and computational

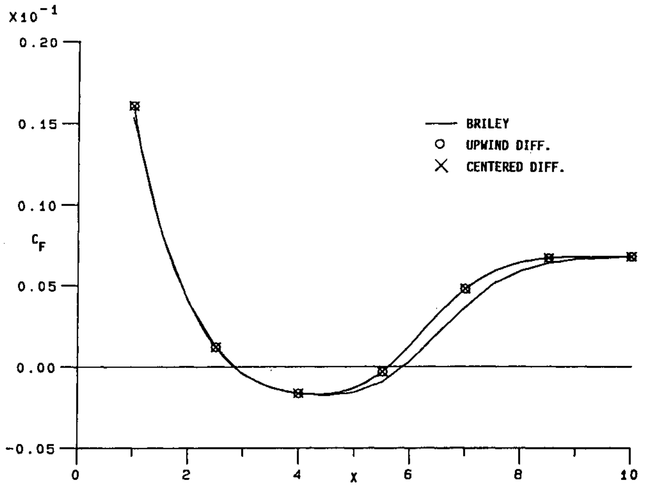


Fig. 2 Effect of discretization method on skin-friction distribution for separated flat flow. Uniform 61 × 31 grid and R = 1024.

efficiency gives similar results as for the external flow problem. The combination of line relaxation of the coupled equations and direct solution of the stream-function equation (Method III) turns out to be about two times faster than the Newton's method with direct solution of both stream-function and vorticity equations (Method IV) and approximately six times faster than line relaxation of the coupled equations only (Method I). Again, the different schemes generate virtually identical solutions.

Conclusions

Using Newton's linearization and Gaussian elimination with partial pivoting to solve the finite-difference equations of the coupled system for ψ and ω simultaneously results in a very robust algorithm, but it requires large storage. However, for two-dimensional problems, this storage requirement is affordable even on a VAX computer. This fully implicit calculation converges quadratically, provided a meaningful initial guess is used. The algorithm is particularly attractive if unstructured grids (finite volumes or finite elements) are used. (For unstructured grids the full Navier-Stokes equations are needed.) Applications of Newton's method and direct solver to three-dimensional problems in general are not possible on the present computers.

The combination of a direct solver for the stream-function equation and line-relaxation method for the coupled stream-function and vorticity equations results in the fastest scheme. This hybrid method requires less memory than the solution technique based on Newton's method and direct solution of ψ and ω simultaneously. The *LU* decomposition of the linear stream-function discrete equations is performed once. Therefore, the subsequent calculations are very inexpensive. Extension of this hybrid scheme for solving three-dimensional flow problems is promising.

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Transonic Flows with Vorticity Transport Around Slender Bodies

Goetz H. Klopfer* and David Nixon†
Nielsen Engineering & Research, Inc.
Mountain View, California

I. Introduction

IN the last decade, methods of modeling vortical incompressible flows around slender bodies have been developed. A review of these methods is given in Ref. 1. The basic premise of these methods is that discrete vortices are introduced into an otherwise potential flow. If the discrete vortices are excluded from the domain, then the flow velocities in the domain can be considered as the superposition of potential flow velocities and the velocities induced by the vortices. The formulation requires that the governing equations be linear (to allow superposition) and, hence, excludes nonlinear compressible flows such as transonic flow. Also the vortex elements must be tracked, and this can become a complicated computational procedure.

This Note is concerned with the derivation of a technique for compressible flows that is similar to that for incompressible flows. The vorticity transport equations are derived from Crocco's equation, and for slender bodies it is found that the flow is isentropic to a first approximation and that only the crossflow vorticity is significant. The latter result is similar to one used in the incompressible theory. The present theory does not require discrete vortices but computes a vorticity field, thus avoiding the need for tracking the vortex elements. In the incompressible limit the "standard" formulation is recovered, and hence the present theory can be regarded as a unifying theory for all speed ranges.

Analysis

The Euler equations for steady compressible flow are

$$(\rho U)_x + (\rho V)_y + (\rho W)_z = 0 \tag{1}$$

$$(\rho U^2 + p)_x + (\rho UV)_y + (\rho UW)_z = 0 \tag{2}$$

$$(\rho UV)_x + (\rho V^2 + p)_y + (\rho VW)_z = 0 \tag{3}$$

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*Research Scientist. Member AIAA.

†President.